Hyper-reduction of mechanical models involving internal variables

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The main illustration of the talk:
A polycrystal plastic model of a specimen in duplex stainless steel

We seek a model reduction method that can take into account recent research results in mechanics of materials.

We expect easier construction of response surfaces by using Reduced Order Models. Here the vector of parameters is $\mu = \{U_{\text{max}}, U_{a}, \Delta U\}$. 
The main illustration of the talk: 
A polycrystal plastic model of a specimen in duplex stainless steel 
[Meric & Cailletaud 94]

The constitutive equations in the framework of continuum thermomechanics of solids: internal variables are introduced to model the history of the material changes.

\[
w(\varepsilon, \varepsilon^p, \alpha^1, ... \alpha^r, ... \rho^1, ... \rho^s, ...) = w^e(\varepsilon - \varepsilon^p) + \frac{1}{2} c \sum_r (\alpha^r)^2 + \frac{1}{2} Q \sum_r \sum_s h_{rs} \rho^r \rho^s \quad (1)
\]

\[
x^r = c \alpha^r \quad r^f = Q \sum_s h_{rs} \rho_s \quad \text{(Taylor model : } h_{rs} = 1) \quad (2)
\]

\[
f^r = |\tau^r - x^r| - r^f - \tau_0 \quad \tau^r = \sigma : \frac{1}{2} (I^f \otimes n^f + n^f \otimes I^f) \quad (3)
\]

\[
Z = \frac{K}{n+1} \sum_r \left( \frac{f^r}{K} \right)^{n+1} \quad (4)
\]

\[
\dot{\varepsilon}^p = \sum_r \dot{\gamma}^r \frac{1}{2} (I^f \otimes n^r + n^f \otimes I^r) \quad \dot{\gamma}^r = \left( \frac{f^r}{K} \right)^n \text{sign}(\tau^r - x^r) \quad (5)
\]

\[
\dot{\alpha}^r = \left( \frac{f^r}{K} \right)^n \left( \text{sign}(\tau^r - x^r) - d \alpha^r \right) \quad \dot{\rho}^r = \left( \frac{f^r}{K} \right)^n (1 - b \rho^r) \quad (6)
\]

There is 18 internal variables at each material point. The computational complexity of the constitutive equations is not negligible compared to the complexity of the equilibrium equation.
The main goal of model reduction methods is the design of surrogate models having few state variables, involving few equilibrium equations, while keeping the same material parameters.

**Issues:**
What can be the contribution of model reduction methods to nonlinear homogenization?
Is it possible to split the set of internal variables into a set of explanatory variables and a set of explained variables?
Is it possible to extend the concept of symmetry to define explanatory variables?
Reduced-Basis approximations

We consider two kinds of Reduced-Basis approximations.

- The usual one, constant RB with respect to the parameters \( \mu \in D \) of the model:

\[
\mathbf{u}(x, \mu, t) = \mathbf{u}_o(x, \mu, t) + \sum_{k=1}^{N} \phi_k(x) \varphi_k(t, \mu), \quad x \in \Omega \quad \mu \in D \quad t \in ]0, T],
\]

\[
\phi_k(x) = \sum_{i=1}^{N} N_i(x) A_{ik}, \quad k = 1, \ldots, N
\]

- The multidimensional RB composed of vectors defined over \( \Omega \times D \):

\[
\mathbf{u}(x, \mu, t) = \mathbf{u}_o(x, \mu, t) + \sum_{k=1}^{N} \phi'_k(x, \mu) \varphi'_k(t), \quad x \in \Omega \quad \mu \in D \quad t \in ]0, T],
\]

\[
\phi'_k(x, \mu) = \sum_{i=1}^{N} N_i(x) A'_{ik}(\mu), \quad k = 1, \ldots, N
\]

The Ammar-Chinesta point of view: \( (x, \mu) \) is denoted \( x' \). It is a point of \( \Omega' = \Omega \times D \).

Therefore Equation (37) is similar to Equation (7). Only a finite number of sampling points of the parameter domain are considered: \( \mathcal{F}_D = \{ \mu_p \}^P_{p=1} \). These sampling points are provided by a Design Of Experiment Method (Kriging, Full Factorial design, ...).
Example of multidimensional model in crystal plasticity theory: $\mathcal{D} = \mathbb{R}^3$, $\mathcal{P} = 2^3$

This is a 2D projection of one multidimensional mesh for the simultaneous simulation of 8 cases of loading. There is 434,000 FE dofs and 15.4e6 internal variables.

\[
\begin{align*}
\mathbf{u}(\mathbf{x}, \mu, t) &= \mathbf{u}_o(\mathbf{x}, \mu, t) + \sum_{k=1}^{k=N} \phi_k(\mathbf{x}') \varphi_k(t), \quad \mathbf{x}' \in \Omega' \quad t \in ]0, T] \\
\phi_k(\mathbf{x}, \mu_p) &= \sum_{i=1}^{i=N} \mathbf{N}_i(\mathbf{x}) A_{jk}, \quad j = i + (p - 1) N, \quad k = 1, \ldots, N, \quad p = 1, \ldots, \mathcal{P}, \\
\alpha(\mathbf{x}, \mu, t) &= \sum_{k=1}^{k=N_\alpha} \mathbf{T}_k(\mathbf{x}') b_k(t), \quad \mathbf{x}' \in \Omega' \quad t \in ]0, T]
\end{align*}
\]  

Here $\alpha$ are the internal variables: $\alpha \in \mathbb{R}^{18}$. The matrix $\mathbf{A}$ has 434,000 rows ($N \times \mathcal{P}$).
Few words on complexity in mechanics of solid materials

We consider the parametrized evaluation problem: given a $\mu \in D$, evaluate the response $s(\mu) = \ell(u, \alpha; \mu)$, where $(u, \alpha) \in ((u_o + V) \times W)$ is the solution of the partial differential equation (14) coupled to the constitutive equation (15) and the initial condition (16) bellow.

$$a_{\Omega}(v, u; \alpha; \mu; t) = 0, \quad \forall v \in V \quad \forall t \in ]0, T],$$

$$\frac{\partial \alpha}{\partial t} = g(\alpha; \nabla u; x; \mu; t), \quad \forall x \in \Omega \quad \forall t \in ]0, T],$$

$$\alpha = \alpha_o, \quad \forall x \in \Omega \quad t = 0$$

Here $V$ is a Hilbert space with the associated inner product $(v, u)_{\Omega}$; and $a_{\Omega}$ is a bounded form linear with respect to $v$. We assume that the following property holds: for all $\Omega_Z \subset \Omega$,

$$a_{\Omega}(v, u; \alpha; \mu; t) = a_{\Omega_Z}(v, u; \alpha; \mu; t) + a_{\Omega \setminus \Omega_Z}(v, u; \alpha; \mu; t)$$

In case of multidimensional problems $\Omega'$ and $L^2(D; V)$ are substituted for $\Omega$ and $V$ respectively.

Equation (15) can be very complex. For instance, its contribution to the computational time of the given example is 33%. Substitute $V_{\text{ROM}} \subset V$ for $V$, has no effect on this complexity even if the dimension of $V_{\text{ROM}}$ is very small. Reduced Order Model may be inefficient in such a case.

The Hyper-Reduction method can overcome this issue.
Outline:

- What is hyper-reduction?
- A theoretical example in case of symmetry.
- Complexity analysis of Hyper-reduced formulations.
- Response surface enrichment by adding a sampling point.
- Response surface enrichment by adding a new dimension to the parameter domain.
- Incremental construction of Hyper-Reduced Model (HRM) using the APHR method (A Priori Hyper-Reduction).
- Various examples of Hyper-Reduced Models.
What is Hyper-Reduction [Ryckelynck 2005, 2009]?

The Hyper-Reduction method aims to write few governing equations having a low complexity by introducing a Reduced Integration Domain $\Omega_Z \subset \Omega$ and related truncated test functions denoted $\{\psi_k\}_{k=1}^N$. The RID is designed by using the shape of the empirical modes of the ROM. The subset of truncated test functions are empirical modes set to zero outside the RID:

$$\psi_k(x) \in \mathcal{V}$$

$$\psi_k(x) = \beta(x) \phi_k \quad \forall x \in \Omega$$

$$\beta(x) = 0 \quad \forall x \in \Omega \setminus \Omega_Z$$

$\beta(x)$ is a given bubble function equal to 1 almost everywhere in $\Omega_Z$. Therefore we introduce the following Petrov-Galerkin formulation: find $u \in (u_o + \text{span}(\phi_k)_{k=1}^N)$ and $\alpha \in \text{span}(\Upsilon_k)_{k=1}^{N_\alpha}$ such that,

$$a_\Omega(\psi_j, u; \alpha; \mu; t) = 0, \quad \forall j \in \{1, ..., N\} \quad \forall t \in [0, T],$$

$$\frac{\partial \hat{\alpha}}{\partial t} = g(\hat{\alpha}; \nabla u; x; \mu; t), \quad \forall x \in \Omega_Z \quad \forall t \in [0, T],$$

$$\hat{\alpha} = \alpha_o, \quad \forall x \in \Omega_Z \quad t = 0,$$

$$\left\{b_k\right\}_{k=1}^{N_\alpha} = \arg \min_{\left\{y_k\right\}_{k=1}^{N_\alpha}} \left\|\hat{\alpha} - \sum_{k=1}^{k=N_\alpha} \Upsilon_k y_k\right\|_{\Omega_Z}^2,$$

Property : $a_\Omega(\psi_j, u; \alpha; \mu; t) = a_{\Omega_Z}(\psi_j, u; \alpha; \mu; t)$, the smaller $\Omega_Z$ the smaller the complexity.
Property: the Hyper-reduction of symmetrical problems can be exact

Several Hyper-reduced formulations of a symmetrical problem can be proposed such that the solution is unchanged.

\[ \mathbf{x} = r \mathbf{e}_r(\theta) + z \mathbf{e}_3 \quad \Omega = \bar{\Omega} \times [0, 2\Pi] \]

\[ \beta(\theta) = 0 \text{ over } [-\Pi, -\delta\theta] \cup [\delta\theta, \Pi] \]

\[ \mathbf{u} \in \mathcal{V}_{ROM} = \{ \mathbf{v} \in \mathcal{V} | \frac{\partial \mathbf{v}}{\partial \theta} = 0 \} \]

\[ \mathcal{V}_{ZROM} = \{ \mathbf{v} \in \mathcal{V} | \mathbf{v} = \beta \mathbf{w}, \text{ with } \mathbf{w} \in \mathcal{V}_{ROM} \} \]

The 3D Principle of Virtual Work using truncated test functions is equivalent to the classical 2D weak formulation. In a sense, the Hyper-Reduction is an extension of the symmetry concept.
Proof: the Hyper-reduction of symmetrical problems can be exact

\[ \nabla \mathbf{v} = \beta \nabla \mathbf{w} + \frac{d \beta}{d \theta} f(r, z) \]  \hspace{1cm} (25)

\[ a_{\Omega} (\mathbf{v}, \mathbf{u}; \alpha; \mu; t) = \int_{\Omega} \nabla \mathbf{v} E(r, z) \nabla \mathbf{v} d\Omega - \int_{\partial F \Omega} \mathbf{v} \mathbf{F} dS \hspace{0.5cm} \mathbf{v} \in \mathcal{V}_{Z ROM} \]  \hspace{1cm} (26)

\[ = \int_{-\delta \theta}^{\delta \theta} \beta d\theta \left( \int_{\Omega} \nabla \mathbf{w} E(r, z) \nabla \mathbf{u} r dr dz - \int_{\partial F \Omega} \mathbf{w} \mathbf{F} dS \right) \]

\[ + \int_{-\delta \theta}^{\delta \theta} \frac{d \beta}{d \theta} d\theta \left( \int_{\Omega} \ldots dr dz \right) \]  \hspace{1cm} (27)

\[ = \int_{-\delta \theta}^{\delta \theta} \beta d\theta \left( \int_{\Omega} \nabla \mathbf{w} E(r, z) \nabla \mathbf{u} r dr dz - \int_{\partial F \Omega} \mathbf{w} \mathbf{F} dS \right) \]  \hspace{1cm} (28)

If \( \int_{-\delta \theta}^{\delta \theta} \beta d\theta \neq 0 \), then \( a_{\Omega} (\mathbf{v}, \mathbf{u}; \alpha; \mu; t) = 0 \hspace{0.5cm} \forall \mathbf{v} \in \mathcal{V}_{Z ROM} \) is equivalent to:

\[ \int_{\Omega} \nabla \mathbf{w} E(r, z) \nabla \mathbf{u} r dr dz - \int_{\partial F \Omega} \mathbf{w} \mathbf{F} dS = 0 \hspace{0.5cm} \forall \mathbf{w} \in \mathcal{V}_{ROM} \]
Few comments on the Hyper-Reduction method

- The internal variables inside \( \Omega_Z \) are explanatory variables. The internal variables outside \( \Omega_Z \) are explained variables. A similar partition of the variables has been proposed in the Partial Least Square (PLS) method [Wold & Herman 1966].

\[
\{ b_k \}_{k=1}^{N_\alpha} = \arg \min_{\{ y_k \}_{k=1}^{N_\alpha}} \left\| \hat{\alpha} - \sum_{k=1}^{k=N_\alpha} \gamma_k y_k \right\|_{\Omega_Z}
\]

- A minimization similar to Equation (24) was also introduced in the Gappy POD [Everson & Sirovich 1995] by using a mask similar to \( \beta \). But the Hyper-reduction does not aim to create a RB by using gappy data.

- The truncated test functions are obtained by using a rectangular truncation matrix \( Z \) \((m \times N)\):

\[
\psi_k = \sum_{i=1}^{N} N_i(x) \beta_{ii} A_{ik}, \quad (29)
\]

\[
\beta = Z^T Z \quad (30)
\]

- The Hyper-Reduction reduces the complexity of the assembly procedure related to the linearized equilibrium equation \((J \delta q = -R)\):

\[
A^T Z^T Z J A \delta \phi = -A^T Z^T Z R \quad (31)
\]

- Truncated products for RB can also be found in the EIM method [Barrault & al. 2004].

- The Hyper-Reduction of the linearized problem does not require any "off-line" computation. This facilitates adaptive procedures.
An heuristic construction of the RID $\Omega_Z$

$\Omega_Z$ is built by aggregating balls whose the radius is $h/2$ or $2h$. Each vector of reduced bases provides a center of a ball.

- for each mode $\mathbf{\Upsilon}_k$ related to an internal variable $\alpha$ the center of the ball is:

$$x = \arg \max_{y \in \Omega} |\mathbf{\Upsilon}_k(y)|$$

(32)

- for each displacement mode $\phi_k$ the center of the ball is:

$$x = \arg \max_{y \in \Omega} (\varepsilon(\phi_k)(y) : \varepsilon(\phi_k)(y))$$

(33)

One can add to the RID a zone of interest where we need the stress to be estimated. Balls having a radius equal to $2h$ are related to the displacement modes to facilitate the selection of equilibrium conditions. A region covering a part of the loaded boundary is also added to the RID to avoid the spurious solution $u = 0$. 

Hyper-reduction of nonlinear mechanical models Wokshop RB/PDG/POD ENS Cachan November 2011 13/29
Example of RID construction

\[ \arg \max_{Y_k \in \Omega} \| Y_k \| \]

\[ Y_1(\gamma_5) \quad Y_2(\gamma_5) \quad Y_{10}(\gamma_5) \]

One element of the RID

\[ \arg \max_{\varepsilon(\phi_k) : \varepsilon(\phi_k)} \]

\[ \phi_1 \quad \phi_2 \quad \phi_7 \]

An element and its neighbour elements

There is about 1800 empirical modes related to A.
The selection of a set of FE equilibrium equations

The truncation matrix $\mathbf{Z}$ is deduced from the shape of $\Omega_Z$. We take into account the FE equilibrium equations that we are able to write knowing that the stress tensor is computed only over $\Omega_Z$.

The FE equilibrium equations that we can’t take into account are related to the degrees-of-freedoms connected to $\Omega/\Omega_Z$. 

\[
\beta = 0 \text{ (over } \Omega/\Omega_Z) \\
\beta = 1, Z_{ji} = 1
\]
Indicators of computational complexity reduction

Let's $m$ be the number of $Z$ rows, $\omega$ the average number of non zero entries per row of $J$.

$$A^T Z^T Z J A \delta \varphi = -A^T Z^T Z R$$  \hspace{1cm} (34)

The left hand side term requires $2 \omega m N + 2 m N^2$ flops and the right hand side term $2 N m$. The usual Galerking procedure is obtained with $\Omega_Z = \Omega$ and $Z^T Z = I_d$ and the related number of fops are $2 \omega \mathcal{N} N + 2 \mathcal{N} N^2$ and $2 N \mathcal{N}$.

Therefore the related complexity reduction factor is :

$$G_Z = \frac{\mathcal{N}}{m}$$

In case of multidimensional analyses this factor is :

$$G'_Z = \frac{\mathcal{N} P}{m}$$

An other complexity reduction factor related to the integration of the constitutive equations reads:

$$G_I = \frac{\text{Number of elements in } \Omega}{\text{Number of elements in } \Omega_Z}$$
Response Surface enrichment: by adding a new sampling point denoted $\mu_{P+1}$.

The Reduced Bases are provided by the snapshot POD applied to the interpolated state given by the multidimensional model such that:

\[
\sum_{k=1}^{N} \phi_k(x) \varphi_k^*(t) = \sum_{k=1}^{N'} \phi_k'(x, \mu_{P+1}) \varphi_k'(t), \quad x \in \Omega \quad t \in [0, T],
\]

\[
\sum_{k=1}^{N_{\alpha}} \gamma_k(x) b_k^*(t) = \sum_{k=1}^{N'_{\alpha}} \gamma_k'(x, \mu_{P+1}) b_k'(t), \quad x \in \Omega \quad t \in [0, T],
\]

The updated reduced variables $\{\phi_k\}_{k=1}^{N}$ and $\{b_k\}_{k=1}^{N_{\alpha}}$ are provided by an Hyper-Reduced simulation.

Figure: 3rd sliding system internal variable.

Error < 2.5%, $G_Z = 149$, $G_I = 25$, Speed up = 17.7 (Response Surface speed up > 1000). All the internal variables of the reference model are estimated locally.
Numerical results: The RID

Full domain

54240 dof
17782 elements, $2 \times 10^6$ internal variables

Reduced Integration Domain obtained by the Hyper-Reduction method

7 empirical modes, 7 equilibrium equations
705 elements, 76140 internal variables ($1/25$)

Error $< 2.5\%$, $G_Z = 149$, $G_I = 25$, Speed up $= 17.7$.

There is $76,000$ explanatory variables (inside the RID) and $1,800,000$ explained internal variables.
Response Surface enrichment: by adding a new dimension to the parameter domain

ANR MELOXEL project: pitting corrosion of a duplex stainless steel
V. Vignal, H. Pelletier, G. Cailletaud, D. M. Benziane, D. Ryckelynck

Here the former vector of parameters is $\mu = \{U_{\text{max}}, U_a, \Delta U\}$ and the additional parameter is $\nu = \max_t u_{c2}(t)$. 
Response Surface enrichment: by adding a new dimension to the parameter domain

The new parameter domain is $\mathcal{D} \times \nu$.

\[
\begin{align*}
\mathbf{u}(\mathbf{x}, \mu, t, \nu) &= \mathbf{u}_o(\mathbf{x}, \mu, t, \nu) + \sum_{k=1}^{N} \phi'_k(\mathbf{x}, \mu) \phi'_k(t, \nu), \quad \mathbf{x} \in \Omega \quad \mu \in \mathcal{D} \quad t \in [0, T], \\
\phi'_k(\mathbf{x}, \mu) &= \sum_{i=1}^{N} \mathbf{N}_i(\mathbf{x}) A'_{ik}(\mu), \quad k = 1, \ldots, N \\
\alpha'(\mathbf{x}, \mu, t, \nu) &= \sum_{k=1}^{N_{\alpha}} \Upsilon'_k(\mathbf{x}, \mu_{P+1}) b'_k(t, \nu)
\end{align*}
\] (37)

The updated reduced variables $\{\phi'_k\}_{k=1}^{N}$ and $\{b'_k\}_{k=1}^{N_{\alpha}}$ are provided by an Hyper-Reduced simulation.

Error < 30\%, $G_Z = 38$, $G_I = 108$, Speed up = 54 (no RS available).
Whatever is the accuracy of the response surface prediction, the related HROM prediction provides a physically-based error indicator. It is the norm of the truncated equilibrium residual:

\[
(v, R(u))_\Omega = a_\Omega (v, u; \alpha; \mu; t), \forall v \in V_{Zh}, \text{with } \frac{\partial \alpha}{\partial t} = g(\alpha; \nabla u; x; \mu; t) \tag{40}
\]

The definition of the error indicator, denoted \( \eta \), reads:

\[
\eta(u, \alpha, t) = \max_{v \in V_{Zh}} \frac{|(v, R(u))_\Omega|}{\sqrt{(v, v)_\Omega}}, \ u \in u_o + V_h \tag{41}
\]

Here:

\[
V_{Zh} = \text{span}(N_i \beta_{ii})_{i=1}^N \tag{42}
\]
The APHR method [Ryckelynck 2005, 2010]

The APHR method provides an incremental adaptive algorithm to create or update the HROM.

\[
A^{(0)}
\]

\[
A^{(n)} \rightarrow A^{(n+1)}
\]

Just a HROM prediction

if \( \eta(t_{i+1}) > \varepsilon_R \), then
find \( \delta u \) in \( V_h \)
and expand the RB

\[
\text{Just a HROM prediction}
\]

The correction step reads: find \( \delta u \) and \( \delta \alpha \) such that:

\[
a_{\Omega}(N_i, u_{HROM} + \delta u; \alpha_{HROM} + \delta \alpha; \mu; t) = 0, \; i = 1, \ldots, N, \quad (43)
\]

\[
\frac{\partial \alpha_{HROM} + \delta \alpha}{\partial t} = g(\alpha_{HROM} + \delta \alpha; \nabla u_{HROM} + \nabla \delta u; x; \mu; t) \quad \forall x \in \Omega, \quad (44)
\]

\[
\alpha_{HROM} + \delta \alpha = \alpha_o \quad \forall x \in \Omega, \; t = 0, \quad (45)
\]

\[
\sum_{k=1}^{N^{(n+1)}} \phi_{k}^{(n+1)} \varphi_{k}^{(n+1)} = \sum_{k=1}^{N^{(n)}} \phi_{k}^{(n)} \varphi_{k}^{(n)} + \delta u, \quad (46)
\]

\[
\sum_{k=1}^{N_{\alpha}^{(n+1)}} \gamma_{k}^{(n+1)} b_{k}^{(n+1)} = \sum_{k=1}^{N_{\alpha}^{(n)}} \gamma_{k}^{(n)} b_{k}^{(n)} + \delta \alpha \quad (47)
\]

The PGD [Ammar, Chinesta & al. 2006] and the LATIN method [Ladeveze 1985] are alternative a priori reduction methods. Mesh and RB adaptation strategy based on the LATIN method can be found in [Pelle & Ryckelynck 2000].
Numerical results: APHR method versus FEM

During the simulation using the APHR method, the reduced approximation has been enriched 26 times (+). After each correction step we obtained points close to the reference stress-strain curve. The smaller $\varepsilon_R$ the smaller is the magnitude of the deviation. The speed up is 2.2.
Numerical results: The time integration, APHR method versus FEM

This illustrates the effect of the adaptation of the reduced approximations during the incremental computation.
Parallel computing is tractable: The Hyper-Reduction provides "green" simulations [Ryckelynck 2010]

The FETI method and the APHR have been combined to generate HROMs on parallel computers.

Here the RID have been set to one subdomain $\Omega_Z = \hat{\Omega}_8$. This provides an amazing speed up: 360. and an energy reduction factor equal to 5,800 by using 1 processor instead of 16.
Other numerical results: The APHR method is not dedicated to homogenization problems

A Damage simulation using the Rousselier model. Unstable problems require ROM adaptivity. [Ryckelynck & al. 2011]
Work in progress: Periodic homogenization (Vivien Courtier)

Periodic formulation:

\[ u(x, t) = E(t) x + w(x, t) \]

with periodic boundary conditions on \( w \)
Conclusion

The APHR method is an a priori approach providing reduced order models and reduced integration domains. The precision and the efficiency of the surrogate model is governed by few parameters.

The Reduced Integration Domain is the support of explanatory internal variables. The internal variables outside the RID are explained using the internal variables of the RID.

By coupling RS model to the Hyper-Reduction method, one can obtain very fast physically-based predictions and error indicators.

An open issue is: Since the Hyper-Reduction method and the APHR method exploit results of previous simulations, how can we use efficiently High Parallel Computing in order to adapt HROM and generate RS?

The parallel simulation of multidimensional problems needs to be improved.
Work in progress

Fatigue, ductile fracture, brittle fracture with plasticity at the crack tip, non-linear damping...

And pattern recognition applied to experimental results.